A PARTICULAR CASE OF THE METHOD OF SMALL PARAMETER IN THE PROBLEM OF SYNCHRONIZATION

PMM Vol. 34, №6, 1970, pp. 1036-1047
M. V. ULITSKAIA
(Leningrad)
(Received April 28, 1970)

Sunchronous motions of objects with one degree of freedom interacting by way of weak constraints are considered in this paper. The initial system of equations with a small parameter is analogous to that studied in Sect. 3 of [1]. Here, however, it is assumed that the equations defining the generating phases and the synchronous frequency constructed from the first order terms are satisfied identically. This leads to a particular case of the small parameter method requiring the higher order approximations to be considered. In this paper synchronous modes in a system of objects have been determined and the necessary and sufficient conditions for their stability obtained. The technical requirements for investigating similar systems are discussed.

In a particular case when the system is conservative in the first approximation, the conditions of existence and stability become identical to those obtained in [1].

1. The basic system. Determination of synchronous solutions. Consider the problem of weak interactions between substantially nonlinear objects, in the absence of external forces, described by the following system with a multidimensional rapidly rotating phase

$$\varphi_k^{\cdot} = \omega_k + \mu X_k^{(1)}(\varphi_k, \ \omega_k) + \mu^2 X_k^{(2)}(\varphi_1,\ldots, \varphi_n, \ \omega_1,\ldots,\omega_n, \mathbf{v}) + \\ + \mu^3 X_k^{(3)}(\varphi_1,\ldots, \varphi_n, \omega_1,\ldots \omega_n, \mathbf{v}) + \mu^4\ldots$$

$$\omega_k^{\cdot} = \mu Y_k^{(1)}(\varphi_k, \omega_k) + \mu^2 Y_k^{(2)}(\varphi_1, \ldots \varphi_n, \omega_1, \ldots, \omega_n, \mathbf{v}) + \\ + \mu^3 Y_k^{(3)}(\varphi_1, \ldots, \varphi_n, \omega_1, \ldots, \omega_n, \mathbf{v}) + \mu^4 \ldots \qquad (k = 1, 2, \ldots, n) \qquad (1.1)$$

$$\mathbf{v} = A\mathbf{v} + \mathbf{F}_1(\varphi_1,\ldots, \varphi_n, \omega_1\ldots, \omega_n) + \mu\, \mathbf{F}_2(\varphi_1,\ldots, \varphi_n, \omega_1,\ldots,\omega_n, \mathbf{v}) + \mu^2\ldots$$

Here φ_k, ω_k ($k = 1, 2,\ldots, n$) denote, respectively, the phase and frequency of the kth isolated object, $\mathbf{v}$ is a N-dimensional coordinate vector of the supporting system and A is a matrix with constant components. All functions in (1.1) are assumed to be 2π-periodic in $\varphi_1,\ldots, \varphi_n$ and $\mu > 0$ is a small parameter. A superscript dot denotes differentiation with respect to time.

System (1.1) is obtained from the equations of the problem dealing with synchronization of almost conservative objects with one degree of freedom [1] by supplementing the equations of motion of the objects by certain terms depending on the partial coordinates and impulses only, and proportional to the quadratic root of the small parameter. The additional terms are assumed to be such that the mean value of one of them obtained for the generating approximation and averaged over a period, is equal to zero.

Physical prerequisites for investigation of a similar system are as follows.

Assume that the synchronized objects are of the mechanical vibrator type. It is necessary that uniformity in the rotation of the asynchronous motor shaft is kept at a sufficiently small level. The latter requirement is reflected in the assumption that all the terms in the equations of motion describing the moments on the vibrator shaft are small. These moments can be assumed to be of varying order of smallness. It may happen that the mean value of one of the moments obtained in the generating approximation, averaged over a period, is equal to zero. In this case it is possible to obtain a system admitting synchronous solutions by assigning a certain order of smallness μ to all the remaining moments. Such assumptions must be made when dealing with the problem of synchronization of Behrens oscillation exciters. Generalization of such cases yields Eqs. (1.1).

Under certain conditions defined below, the system (1.1) allows a synchronous solution analytic in μ of the form

$$\varphi_k = \varphi_{k0} + \mu\varphi_{k1} + \mu^2\varphi_{k2} + \mu^3\ldots$$

$$\omega_k = \omega_{k0} + \mu\omega_{k1} + \mu^2\omega_{k2} + \mu^3\ldots, \qquad \mathbf{v} = \mathbf{v}_0 + \mu\mathbf{v}_1 + \ldots \tag{1.2}$$

in an interval $0 < \mu < \mu_0$.

Introduce now the dimensionless time $\tau = \nu t$, where ν is the unknown frequency of the synchronous mode assumed to exist in the form of a series $\nu = \nu_0 + \mu\nu_1 + \mu^2\ldots$ Equations (1.1) then become

$$\begin{gathered}
\varphi_k' = \beta\omega_k + \mu\,[\beta X_k^{(1)} - \nu_1\beta^2\omega_k] + \mu^2\,[\beta X_k^{(2)} - \\
- \nu_1\beta^2 X_k^{(1)} - (\nu_2\beta^2 - \nu_1^2\beta^3)\ \omega_k] + \mu^3\,[\beta X_k^{(3)} - \\
- \nu_1\beta^2 X_k^{(2)} - (\nu_2\beta^2 - \nu_1^2\beta^3) X_k^{(1)} - (\nu_1^3\beta^4 - 2\nu_1\nu_2\beta^3 + \nu_3\beta^2)\,\omega_k] + \ldots \\
\omega_k' = \mu\beta\, Y_k^{(1)} + \mu^2\,[\beta Y_k^{(2)} - \nu_1\beta^2 Y_k^{(1)}] + \\
+ \mu^3\,[\beta Y_k^{(3)} - \nu_1\beta^2 Y_k^{(2)} - (\nu_2\beta^2 - \nu_1^2\beta^3)\, Y_k^{(1)}] + \ldots \\
\mathbf{v}' = \beta\,(A\mathbf{v} + \mathbf{F}_1) + \mu\,[\beta\,\mathbf{F}_2 - \nu_1\beta^2\,(A\,\mathbf{v} + \mathbf{F}_1)] + \ldots
\end{gathered} \tag{1.3}$$

where

$$\beta = 1/\nu_0$$

We find now the synchronous solution of (1.3) in the form of (1.2). The generating system admits an n-parameter family of solutions of the form

$$\varphi_{k0} = \tau + \alpha_k, \quad \omega_{k0} = \nu_{0k}, \quad \mathbf{v}_0 = \mathbf{v}_0\,(\tau, \alpha_1, \ldots, \alpha_n, \nu_0, \nu_{01}, \ldots, \nu_{0n})$$

where $\mathbf{v}_0$ is a solution of the last equation of (1.3) 2π-periodic in τ with $\mu = 0$ and $\nu_{01} = \nu_{02} = \ldots = \nu_{0n} = \nu_0$; $\alpha_1, \ldots, \alpha_n$ are the generating phase shifts. It is also assumed that the eigenvalues $\varkappa$ of the system

$$\varkappa\mathbf{w} = \beta A\,\mathbf{w} \tag{1.4}$$

lie outside the circles with centers at in (n is an integer) and with a radius of the order of μ. This is equivalent to the assumption that resonance with respect to coordinates $\mathbf{v}$ is absent.

The first approximation system has the form

$$\begin{gathered}
\varphi_{k1}' = \beta\omega_{k1} + \beta\,(X_k^{(1)})_0 - \beta\nu_1 \\
\omega_{k1}' = \beta\,(Y_k^{(1)})_0 \\
\mathbf{v}_1' = \beta A\mathbf{v}_1 + \beta\mathbf{F}_1 + \beta\,(\mathbf{F}_2)_0 - \beta^2\nu_1 A\mathbf{v}_0 - \beta^2\,\nu_1(\mathbf{F}_1)_0
\end{gathered} \tag{1.5}$$

Here and below the parentheses with the subscript zero denote that the quantities are computed for the generating solution.

As stated above, the case under consideration is that of

$$P_k^{(1)}(\nu_{0k}) = \frac{1}{2\pi}\int_0^{2\pi} (Y_k^{(1)})_0\, d\tau = \langle (Y_k^{(1)})_0 \rangle = 0 \qquad (1.6)$$

Therefore the first approximation to the required synchronous solution is periodic for any value of the generating phases and, that the condition of existence of synchronous modes cannot be obtained by considering the first approximation system. This means that here we are dealing with a particular case of the small parameter method and, unlike in [1], higher approximations must be taken into consideration.

Integrating (1.5) gives

$$\omega_{k1} = \beta U_k + \nu_1 - R_k^{(1)}, \qquad \varphi_{k1} = \beta^2 \Gamma_k + \beta V_k + \alpha_{k1} \qquad (1.7)$$

$$\nu_1 = \nu_1(\alpha_1, \ldots, \alpha_n, \nu_0, \nu_{01}, \ldots, \nu_{0n})$$

$$\nu_{01} = \ldots = \nu_{0n} = \nu_0, \qquad \alpha_{k1} = \mathrm{const}$$

$$\Gamma_k = \int U_k\, d\tau, \qquad V_k = \int [(X_k^{(1)})_0 - R_k^{(1)}]\, d\tau$$

$$U_k = \int (Y_k^{(1)})_0\, d\tau, \qquad \langle \Gamma_k \rangle = \langle V_k \rangle = \langle U_k \rangle = 0 \qquad (1.8)$$

Here ν_1 is a solution, 2π-periodic in τ, of the last equation of (1.5).

Conditions of existence of periodic functions ω_{k2} have the form

$$\left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 (\beta^2 \Gamma_k + \beta V_k + \alpha_{k1}) \right\rangle + \left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\right)_0 (\beta U_k + \nu_1 - R_k^{(1)}) \right\rangle + P_k^{(2)} = 0 \qquad (1.9)$$

where

$$P_k^{(2)} = \langle (Y_k^{(2)})_0 \rangle \qquad (1.10)$$

Using condition (1.6) we obtain

$$\beta^2 \left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 \Gamma_k \right\rangle + \beta \left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 V_k \right\rangle + \beta \left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\right)_0 U_k \right\rangle + P_k^{(2)} = 0 \qquad (1.11)$$

Let us now consider condition (1.11) in greater detail. Using (1.8) and the fact that the functions $(Y_k^{(1)})_0$ and Γ_k are 2π-periodic in $\varphi_1, \ldots, \varphi_n$, i.e. in τ and have no constant terms, we find, on integrating by parts, that Eq. (1.11) can be obtained in the form

$$P_k = -\beta \langle (Y_k^{(1)})_0 (X_k^{(1)})_0 \rangle + \beta \left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\right)_0 U_k \right\rangle + P_k^{(2)} = 0 \qquad (1.12)$$

where we must set $\nu_{01} = \nu_{02} = \ldots = \nu_{0n} = \nu_0$.

This constitutes the conditions of existence of a synchronous mode in an interconnected system of objects. Since the initial equations are autonomous, it follows that (1.12) will only yield ν_0 and $(n-1)$ generating phase differences such as e.g. $\alpha_1 - \alpha_n$, or $\alpha_{n-1} - \alpha_n$, with an arbitrary value assigned to one of α_k.

Let us calculate the second approximations to φ_k and ω_k.

Using relations of the form

$$\left\langle \left(\frac{\partial X_k^{(1)}}{\partial \varphi_k}\right)_0 V_k \right\rangle = R_k^{(1)^2} - \langle (X_k^{(1)})_0^2 \rangle \qquad (1.13)$$

we obtain

$$\varphi_{k2} = \beta^4 \int [L_k + \nu_0 M_k - \langle (L_k + \nu_0 M_k)\rangle]\, d\tau + \\ + \alpha_{k1}\beta^2 U_k + \beta^2 (\nu_1 - R_k^{(1)}) \frac{\partial \Gamma_k}{\partial \nu_k} + \beta^2 \Gamma_k^{(2)} - \beta^3 \nu_1 \Gamma_k + \\ + \beta^3 E_k + \beta F_k + \beta \alpha_{k1} [(X_k^{(1)})_0 - R_k^{(1)}] + \beta V_k^{(2)} + \\ + \beta (\nu_1 - R_k^{(1)}) \frac{\partial V_k}{\partial \nu_k} - \nu_1 \beta^2 \Gamma_k - \beta^2 \nu_1 V_k^{(1)} + \alpha_{k2} \tag{1.14}$$

$$\omega_{k2} = \beta^3 L_k + \beta^2 M_k + \beta \alpha_{k1} (Y_k^{(1)})_0 + \beta (\nu_1 - R_k^{(1)}) \frac{\partial U_k}{\partial \nu_k} + \beta U_k^{(2)} - \beta^2 \nu_1 U_k - \\ - \beta^3 \langle L_k\rangle - \beta^2 \langle M_k\rangle - (\nu_1 - R_k^{(1)}) \frac{\partial R_k^{(1)}}{\partial \nu_k} - \beta \left\langle \left(\frac{\partial X_k^{(1)}}{\partial \omega_k}\right)_0 U_k \right\rangle - \\ - \beta^2 \left\langle \left(\frac{\partial X_k^{(1)}}{\partial \varphi_k}\right)_0 (\Gamma_k + \nu_0 V_k) \right\rangle - R_k^{(2)} + \nu_2 \nu_0$$

where

$$L_k = \int d\tau \left[\left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 (\Gamma_k + \nu_0 V_k) - \left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 (\Gamma_k + \nu_0 V_k) \right\rangle \right]$$

$$M_k = \int d\tau \left[\left(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\right)_0 U_k - \left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\right)_0 U_k \right\rangle \right]$$

$$U_k^{(2)} = \int (Y_k^{(2)})_0\, d\tau, \qquad \langle U_k^{(2)}\rangle = 0, \qquad \Gamma_k^{(2)} = \int U_k^{(2)}\, d\tau \tag{1.15}$$

$$\langle \Gamma_k^{(2)}\rangle = 0$$

$$E_k = \int d\tau \left[\left(\frac{\partial X_k^{(1)}}{\partial \varphi_k}\right)_0 (\Gamma_k + \nu_0 V_k) - \left\langle \left(\frac{\partial X_k^{(1)}}{\partial \varphi_k}\right)_0 (\Gamma_k + \nu_0 V_k) \right\rangle \right]$$

$$F_k = \int d\tau \left[\left(\frac{\partial X_k^{(1)}}{\partial \omega_k}\right)_0 U_k - \left\langle \left(\frac{\partial X_k^{(1)}}{\partial \omega_k}\right)_0 U_k \right\rangle \right]$$

2. Stability of synchronous solutions. The synchronous modes having been found, their stabilities may now be analyzed. For the generating solution the characteristic equation of the system in variations of (1.1) has a $2n$-tuple unique root, i.e. $2n$ characteristic indices in the variational equations are critical [2]. Ultimately the stability of the synchronous solutions (1.2) is defined by the signs of the real parts of the critical characteristic indices, and hence only the latter quantities will be calculated.

Following [1] we eliminate from our variational system the variations of coordinates of the supporting system and seek them in the form

$$\delta \nu = \sum_{k=1}^{n} [\xi_k(\tau, \mu)\, \delta\varphi_k + \eta_k(\tau, \mu)\, \delta\omega_k] \tag{2.1}$$

where ξ_k, η_k $(k = 1, 2, \ldots, n)$ are 2π-periodic vector functions of τ. Seeking ξ_k and η_k in the form of series in μ

$$\xi_k = \xi_k^{(0)} + \mu \xi_k^{(1)} + \mu^2 \ldots, \qquad \eta_k = \eta_k^{(0)} + \mu \eta_k^{(1)} + \mu^2 \ldots \tag{2.2}$$

we obtain [1]

$$\xi_k^{(0)} = \frac{\partial \nu_0}{\partial \alpha_k}, \qquad \eta_k^{(0)} = \frac{\partial \nu_0}{\partial \nu_k} + \zeta_k, \quad \zeta_k = \frac{\partial^2 \nu_0}{\partial \nu_0 \partial \alpha_k} \quad (k = 1, 2, \ldots, n) \tag{2.3}$$

Elimination of the variations $\delta\nu$ in accordance with (2.1) yields a system of $2n$

equations in $\delta\varphi_k$ and $\delta\omega_k$ (which is not written out in full). The characteristic indices whose total number is $2n$ are equal to the critical characteristic indices of the initial equations in variations. The $2n$-tuple root of the characteristic equation of this system has the corresponding elementary, nonsimple quadratic divisors of the generating solution. Therefore [3] the critical indices can be expanded either in whole powers of $\mu^{1/2}$, or in whole powers of μ.

Direct computation of the coefficients accompanying $\mu^{1/2}$, $\mu^{3/2}$, $\mu^{5/2}$ gives their values equal to zero. It can further be shown that all the remaining terms containing fractional powers also vanish. The proof is particularly simple when the initial system is conservative to within the terms of the order of μ^2.

Furthermore, by a simple change of variable this system can be reduced to another type of system discussed in [1]

$$\begin{aligned}
\varphi_i^{*\cdot} &= \omega_i^* + \mu^2 [X_{i1}^* (\varphi_i^*,\ldots, \varphi_n^*, \omega_1^*,\ldots, \omega_n^*, \mathbf{v}) + \\
&+ \mu X_{i2}^* (\varphi_1^*,\ldots, \varphi_n^*, \omega_1^*,\ldots, \omega_n^*, \mathbf{v})] + \mu^4\ldots \\
\omega_i^{*\cdot} &= \mu^2 [Y_{i1}^* (\varphi_1^*,\ldots, \varphi_n^*, \omega_1^*,\ldots, \omega_n^*, \mathbf{v}) + \\
&+ \mu Y_{i2}^* (\varphi_1^*,\ldots, \varphi_n^*, \omega_1^*,\ldots, \omega_n^*, \mathbf{v})] + \mu^4\ldots \\
\mathbf{v}^{\cdot} &= A\mathbf{v} + \mathbf{F}^* (\varphi_1^*,\ldots, \varphi_n^*, \omega_1^*,\ldots, \omega_n^*) + \mu\ldots
\end{aligned} \tag{2.4}$$

from which it follows at once that in this case the indices are expanded in the powers of μ.

Let now the substitution

$$\delta\varphi_k = e^{\lambda\tau}\vartheta_k, \qquad \delta\omega_k = e^{\lambda\tau}\psi_k \qquad (k = 1, 2, \ldots, n) \tag{2.5}$$

be made, where λ is the required critical characteristic index, which is sought in the form

$$\lambda = \lambda_1\mu + \lambda_2\mu^2 + \lambda_3\mu^3 + \ldots \tag{2.6}$$

The quantities $\lambda_1, \lambda_2, \ldots$ are obtained from the condition of existence of a 2π-periodic solution of a system obtained by making the substitution (2.5) in the variational equations. The solutions are sought in the form of series

$$\begin{aligned}
\vartheta_k &= \vartheta_k^{(0)} + \mu\vartheta_k^{(1)} + \mu^2\vartheta_k^{(2)} + \mu^3\ldots \\
\psi_k &= \psi_k^{(0)} + \mu\psi_k^{(1)} + \mu^2\psi_k^{(2)} + \mu^3\ldots
\end{aligned} \tag{2.7}$$

where $\vartheta_k^{(i)}$, $\psi_k^{(i)}$ $(i = 0, 1, 2,\ldots)$ are 2π-periodic functions of τ.

In the zeroth approximation we have

$$\psi_k^{(0)} = 0, \qquad \vartheta_k^{(0)} = a_k \qquad (a_k = \text{const}) \tag{2.8}$$

Considering the terms of order of μ, we obtain the following system of equations for $\vartheta_k^{(1)}$ and $\psi_k^{(1)}$:

$$\vartheta_k^{(1)\prime} = \beta\psi_k^{(1)} + \beta\left(\frac{\partial X_k^{(1)}}{\partial\varphi_k}\right)_0 a_k - \lambda_1 a_k, \qquad \psi_k^{(1)\prime} = \beta\left(\frac{\partial Y_k^{(1)}}{\partial\varphi_k}\right)_0 a_k \tag{2.9}$$

In accordance with condition (1.6), the functions $\psi_k^{(1)}$ $(k = 1, 2,\ldots, n)$ obtained from the second set of n equations of (2.9) are periodic functions. Integration of the second set of equations from (2.9) yields

$$\psi_k^{(1)} = \beta a_k (Y_k^{(1)})_0 + b_{*k} \qquad (b_{*k} = \text{const}) \tag{2.10}$$

The constants b_{*k} are obtained from the condition of periodicity of the functions $\vartheta_k^{(1)}$: $b_{*k} = \nu_0\lambda_1 a_k$. The functions $\vartheta_k^{(1)}$ are now obtained from the first set of n

equations of (2.9)

$$\vartheta_k^{(1)} = \beta^2 a_k U_k + \beta (X_k^{(1)})_0 a_k + b_k \qquad (b_k = \text{const}) \tag{2.11}$$

The second approximation system has the form

$$\begin{aligned}
\vartheta_k^{(2)\prime} &= \beta\psi_k^{(2)} - \lambda_1 [\beta^2 a_k U_k + \beta (X_k^{(1)})_0 a_k + b_k] + \\
&+ \beta \left(\frac{\partial X_k^{(1)}}{\partial \varphi_k}\right)_0 [\beta^2 a_k U_k + \beta a_k (X_k^{(1)})_0 + b_k] + \beta \left(\frac{\partial X_k^{(1)}}{\partial \omega_k}\right)_0 \times \\
&\times [\beta a_k (Y_k^{(1)})_0 + \lambda_1 a_k \nu_0] - \nu_1 \beta^2 [\beta a_k (Y_k^{(1)})_0 + \lambda_1 a_k \nu_0] - \\
&- \lambda_2 a_k + \beta \sum_{j=1}^{n} \left(\frac{\partial X_k^{(2)}}{\partial \varphi_j}\right)_0 a_j + \beta \left(\frac{\partial X_k^{(2)}}{\partial \nu}\right)_0 \sum_{j=1}^{n} \xi_j a_j - \\
&- \nu_1 \beta^2 \left(\frac{\partial X_k^{(1)}}{\partial \varphi_k}\right)_0 a_k + \beta \left(\frac{\partial^2 X_k^{(1)}}{\partial \varphi_k^2}\right)_0 \varphi_{k1} a_k + \beta \left(\frac{\partial^2 X_k^{(1)}}{\partial \varphi_k \partial \omega_k}\right)_0 \omega_{k1} a_k
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
\psi_k^{(2)\prime} &= -\lambda_1 \beta a_k (Y_k^{(1)})_0 - \lambda_1^2 a_k \nu_0 + \beta \left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 [b_k + \beta^2 a_k U_k + \beta (X_k^{(1)})_0 a_k] + \\
&+ \beta \left(\frac{\partial X_k^{(1)}}{\partial \omega_k}\right)_0 [\beta a_k (Y_k^{(1)})_0 + \lambda_1 a_k \nu_0] + \\
&+ \beta \sum_{j=1}^{n} \left(\frac{\partial Y_k^{(2)}}{\partial \varphi_j}\right)_0 a_j + \beta \left(\frac{\partial Y_k^{(2)}}{\partial \nu}\right)_0 \sum_{j=1}^{n} \xi_j a_j - \\
&- \beta^2 \nu_1 \left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 a_k + \beta \left(\frac{\partial^2 Y_k^{(1)}}{\partial \varphi_k^2}\right)_0 \varphi_{k1} a_k + \beta \omega_{k1} a_k \left(\frac{\partial^2 Y_k^{(1)}}{\partial \varphi_k \partial \omega_k}\right)_0
\end{aligned}$$

Using the relations

$$\begin{gathered}
\left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 U_k \right\rangle + \left\langle \left(\frac{\partial^2 Y_k^{(1)}}{\partial \varphi_k^2}\right)_0 \Gamma_k \right\rangle = 0 \\
\left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 (X_k^{(1)})_0 \right\rangle + \left\langle \left(\frac{\partial^2 Y_k^{(1)}}{\partial \varphi_k^2}\right)_0 V_k \right\rangle = 0 \\
\left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\right)_0 (Y_k^{(1)})_0 \right\rangle + \left\langle \left(\frac{\partial^2 Y_k^{(1)}}{\partial \varphi_k \partial \omega_k}\right)_0 U_k \right\rangle = 0
\end{gathered} \tag{2.13}$$

we obtain, from the condition of periodicity of the second approximation, the following system of equations for determining a_i:

$$\begin{gathered}
\sum_{j=1}^{n} \left(\frac{\partial P_k^{(2)}}{\partial \alpha_j} - \lambda_1^2 \nu_0^2 \delta_{kj}\right) a_j = 0 \qquad (k = 1, 2, \ldots, n) \\
P_k^{(2)} = P_k^{(2)} (\alpha_1, \ldots, \alpha_n, \nu_0, \nu_{01}, \ldots, \nu_{0n}) = \langle (Y_k^{(2)})_0 \rangle \\
\nu_{01} = \nu_{02} = \ldots = \nu_{0n} = \nu_0
\end{gathered} \tag{2.14}$$

Thus the roots of the equation

$$\left| \frac{\partial P_k^{(2)}}{\partial \alpha_j} - \lambda_1^2 \delta_{kj} \nu_0^2 \right| = 0 \tag{2.15}$$

represent the first approximations to the characteristic indices.

From the fact that the initial system (1.1) is autonomous it follows that Eq. (2.15) regarded as an nth degree equation in λ_1^2 has a zero root. The necessary condition of stability (for the first group) is, that all the remaining roots $(\lambda_1^2)_1, \ldots, (\lambda_1^2)_{n-1}$ are negative (the case of a multiple zero root is not considered). The synchronous mode is unstable even if a single positive or complex root exists. Suppose the roots are negative. Then $(2n - 2)$ critical indices of the variational system are represented by the following expansions:

$$\lambda_k^{(1)} = i\mu\, |(\lambda_1^2)_k|^{1/2} + \mu^2\ldots, \quad \lambda_k^{(2)} = -i\mu\, |(\lambda_1^2)_k|^{1/2} + \mu^2\ldots \qquad (k = 1, 2, \ldots, n-1)$$

one index will be given by $\lambda_n^{(1)} = \lambda_{2n}\mu^2 + \ldots,$ and the remaining one will be zero. This shows that if the conditions of stability for the first group hold, the stability of the system is substantially influenced by the μ^2-order terms appearing in the expansions of the unknown indices which can now be computed.

Subsequently we will need the second approximations $\psi_k^{(2)}$ and $\vartheta_k^{(2)}$. Using the relations

$$\left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 U_k + \left(\frac{\partial^2 Y_k^{(1)}}{\partial \varphi_k^2}\right)_0 \Gamma_k = \frac{\partial}{\partial \alpha_k}\left[\left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 \Gamma_k\right]$$

$$\left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 (X_k^{(1)})_0 + \left(\frac{\partial^2 Y_k^{(1)}}{\partial \varphi_k^2}\right)_0 V_k = \frac{\partial}{\partial \alpha_k}\left[\left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 V_k\right] + R_k^{(1)}\left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0$$

$$\left(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\right)_0 (Y_k^{(1)})_0 + \left(\frac{\partial^2 Y_k^{(1)}}{\partial \varphi_k \partial \omega_k}\right)_0 U_k = \frac{\partial}{\partial \alpha_k}\left[\left(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\right)_0 U_k\right] \qquad (2.16)$$

and integrating (2.12) we obtain

$$\psi_k^{(2)} = -\beta\lambda_1 a_k U_k + \beta^3 a_k\left[\left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 \Gamma_k - \right.$$

$$\left. - \left\langle\left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 \Gamma_k\right\rangle\right] + \beta^2 a_k\left[\left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 V_k - \left\langle\left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 V_k\right\rangle\right] +$$

$$+ R_k^{(1)}\beta^2 a_k (Y_k^{(1)})_0 + \beta b_k (Y_k^{(1)})_0 + \beta^2 a_k\left[\left(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\right)_0 U_k - \left\langle\left(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\right)_0 U_k\right\rangle\right] +$$

$$+ \lambda_1 a_k \frac{\partial U_k}{\partial \nu_k} + \beta\sum_{j=1}^{n} a_j \frac{\partial U_k^{(2)}}{\partial \alpha_j} - \nu_1\beta^2 a_k (Y_k^{(1)})_0 + \alpha_{k1}\beta a_k\left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 + \lambda_1 b_k \nu_0 + \qquad (2.17)$$

$$+ (\nu_1 - R_k^{(1)})\beta\left(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\right)_0 a_k - \lambda_1 a_k \frac{\partial R_k^{(1)}}{\partial \nu_k}\nu_0 + \nu_1 a_k \lambda_1 + \lambda_2 a_k \nu_0 - \sum_{j=1}^{n}\frac{\partial R_k^{(2)}}{\partial \alpha_j} a_j$$

$$\vartheta_k^{(2)} = \beta^4 a_k L_k - 2\lambda_1\beta^2 a_k \Gamma_k + R_k^{(1)}(X_k^{(1)})_0 + \beta^3 a_k M_k +$$

$$+ \beta^3 R_k^{(1)} a_k U_k + \beta^2 b_k U_k + \beta\lambda_1 a_k \frac{\partial \Gamma_k}{\partial \nu_k} + \beta^2\sum_{j=1}^{n} a_j \frac{\partial \Gamma_k^{(2)}}{\partial \alpha_j} -$$

$$- \beta^3 \nu_1 a_k U_k + \alpha_{k1}\beta^2 a_k (Y_k^{(1)})_0 + (\nu_1 - R_k^{(1)})\beta^2 a_k \frac{\partial U_k}{\partial \nu_k} +$$

$$+ \beta b_k (X_k^{(1)})_0 + \beta^2 a_k\left(\frac{\partial X_k^{(1)}}{\partial \omega_k}\right)_0 U_k + \beta^3 a_k\left(\frac{\partial X_k^{(1)}}{\partial \varphi_k}\right)_0 \Gamma_k +$$

$$+\lambda_1 a_k \frac{\partial V_k}{\partial \nu_k} + \beta^2 a_k \left(\frac{\partial X_k^{(1)}}{\partial \varphi_k}\right)_0 V_k - \nu_1 \beta^3 a_k U_k +$$

$$+\beta \sum_{j=1}^{n} \frac{\partial V_k^{(2)}}{\partial \alpha_j} a_j - \nu_1 \beta^2 a_k (X_k^{(1)})_0 + \alpha_{k1} \beta a_k \left(\frac{\partial X_k^{(1)}}{\partial \varphi_k}\right)_0 -$$

$$-\lambda_1 a_k \beta \nu_k + \beta(\nu_1 - R_k^{(1)}) \left(\frac{\partial X_k^{(1)}}{\partial \omega_k}\right)_0 a_k + c_k \quad (c_k = \text{const})$$

3. Conditions of stability of the second group. Using the relations

$$\left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 \frac{\partial \Gamma_k}{\partial \nu_k} \right\rangle + \left\langle \left(\frac{\partial^2 Y_k^{(1)}}{\partial \varphi_k \partial \omega_k}\right)_0 \Gamma_k \right\rangle = 0$$

$$\left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \varphi_k}\right)_0 \frac{\partial V_k}{\partial \nu_k} \right\rangle + \left\langle \left(\frac{\partial^2 Y_k^{(1)}}{\partial \varphi_k \partial \omega_k}\right)_0 V_k \right\rangle = -\frac{\partial}{\partial \nu_k} \langle (Y_k^{(1)})_0 (X_k^{(1)})_0 \rangle \quad (3.1)$$

$$\left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\right)_0 \frac{\partial U_k}{\partial \nu_k} \right\rangle + \left\langle \left(\frac{\partial^2 Y_k^{(1)}}{\partial \omega_k^2}\right)_0 U_k \right\rangle = \frac{\partial}{\partial \nu_k} \left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\right)_0 U_k \right\rangle$$

$$\left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\right)_0 (Y_k^{(1)})_0 \right\rangle + \left\langle \left(\frac{\partial^2 Y_k^{(1)}}{\partial \varphi_k \partial \omega_k}\right)_0 U_k \right\rangle = 0$$

let us write out the third approximation system.

The condition of existence of periodic functions $\operatorname{Im} \psi_k^{(3)}$ implies that the imaginary parts of the quantities b_j must satisfy the system

$$\sum_{j=1}^{n} \left[\frac{\partial P_k^{(2)}}{\partial \alpha_j} - \lambda_1^2 \nu_0^2 \delta_{kj}\right] \operatorname{Im} b_j = \quad (3.2)$$

$$= \lambda_1 \left\{ -\beta^2 \langle (Y_k^{(1)})_0 (X_k^{(1)})_0 \rangle a_k + \beta^2 \left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\right)_0 U_k \right\rangle a_k + \right.$$

$$\left. + 2 \operatorname{Re} \lambda_2 \nu_0 a_k - \sum_{j=1}^{n} \left[\frac{\partial P_k}{\partial \nu_j} + \frac{\partial R_k^{(2)}}{\partial \alpha_j} + \left\langle \left(\frac{\partial Y_k^{(2)}}{\partial \nu}\right)_0 \zeta_j \right\rangle \right] a_j \right\}$$

The condition of solvability of the inhomogeneous system (3.2) now yields $(n-1)$ pairs of second approximations corresponding to the nonzero roots of (2.15)

$$\operatorname{Re}\lambda_{2r} = \beta^3 \sum_{k=1}^{n} \left[\langle (Y_k^{(1)})_0 (X_k^{(1)})_0 \rangle - \left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\right)_0 U_k \right\rangle \right] a_k^{(r)} a_k^{(r)*} +$$

$$+ \sum_{k=1}^{n} \sum_{j=1}^{n} \left[\frac{\partial P_k}{\partial \nu_j} + \frac{\partial R_k^{(2)}}{\partial \alpha_j} + \left\langle \left(\frac{\partial Y_k^{(2)}}{\partial \nu}\right)_0 \zeta_j \right\rangle \right] a_j^{(r)} a_k^{(r)*} \quad (3.3)$$

$$(r = 1, 2, \ldots, n-1)$$

Here the indexing of the solutions of (2.14) has been made more precise, namely, $a_k^{(r)}$ means that the solution corresponds to the root $(\lambda_1)_r$ $(r = 1, 2, \ldots, n-1)$. Furthermore, it is assumed that all the roots of (2.15) are simple and that the corresponding vectors $\mathbf{a}^{(r)}$ and $\mathbf{a}^{(r)*}$ are normalized

$$\sum_{k=1}^{n} a_k^{(r)} a_k^{(r)*} = 1 \quad (3.4)$$

The zero roots of (2.16), as was shown before, correspond to the characteristic indices of the complete variational system, the indices given by

$$\lambda_n^{(1)} = \lambda_{2n}\mu^2 + \ldots, \ \lambda_n^{(2)} = 0 \tag{3.5}$$

Let the characteristic index $\lambda_n^{(1)}$ be computed to an accuracy greater than μ^2. The sum of the indices of a system of equations with periodic coefficients is equal to the value of the trace of the coefficient matrix averaged over a period. Computing this quantity for variational system we obtain

$$\sum_{r=1}^{n} (\lambda_r^{(1)} + \lambda_r^{(2)}) = \mu^2 \sum_{k=1}^{n} \Big[\beta \frac{\partial R_k^{(2)}}{\partial \alpha_k} + \beta^3 \langle (X_k^{(1)})_0 (Y_k^{(1)})_0\rangle -$$
$$- \beta^3 \Big\langle \Big(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\Big)_0 U_k \Big\rangle + \beta \frac{\partial P_k}{\partial \nu_k} + \beta \Big\langle \Big(\frac{\partial Y_k^{(2)}}{\partial \nu}\Big)_0 \zeta_k \Big\rangle\Big] + \mu^3 \ldots \tag{3.6}$$

From this, performing transformations analogous to those given in [1] we find that the characteristic index is defined with the required accuracy from the following relation:

$$\lambda_{2n} = \sum_{k,\,j=1}^{n} \Big\{\Big[\beta \frac{\partial P_k}{\partial \nu_j} + \beta \frac{\partial R_k^{(2)}}{\partial \alpha_j} + \beta \Big\langle \Big(\frac{\partial Y_k^{(2)}}{\partial \nu}\Big)_0 \zeta_j \Big\rangle\Big] a_j^{(n)} a_k^{(n)*} +$$
$$+ \Big[\beta^3 \langle (Y_k^{(1)})_0 (X_k^{(1)})_0\rangle - \beta^3 \Big\langle \Big(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\Big)_0 U_k \Big\rangle\Big] a_k^{(n)} a_k^{(n)*}\Big\} \tag{3.7}$$

The conditions of stability of the second group are represented by $\operatorname{Re}\lambda_{2r} < 0$ ($r = 1, 2, \ldots, n$), where $\operatorname{Re}\lambda_{2r}$ is computed according to (3.3).

The investigation of the stability becomes much simpler in the case when the vector function $\mathbf{F}_1$ can be written as a sum

$$\mathbf{F}_1 = \sum_{k=1}^{n} \mathbf{F}_{1k}(\varphi_k, \omega_1, \ldots, \omega_n) \tag{3.8}$$

and the functions $Y_k^{(2)}$ are linear with respect to coordinates of the supporting system

$$Y_k^{(2)} = Y_{k0}^{(2)}(\varphi_1, \ldots, \varphi_n, \omega_1, \ldots, \omega_n) + Y_{k1}^{(2)}(\varphi_1, \ldots, \varphi_n, \omega_1, \ldots, \omega_n)\, \mathbf{v} \tag{3.9}$$

The vector $\mathbf{v}$ in the generating approximation is now of the form of superposition

$$\mathbf{v}_0 = \sum_{k=1}^{n} \mathbf{v}_{0k}(\tau + \alpha_k, \nu_0, \nu_{01}, \ldots, \nu_{0n}) \tag{3.10}$$

where the components $\mathbf{v}_{0k}$ are given by

$$\mathbf{v}_{0k}' = \beta\,[A\,\mathbf{v}_{0k} + \mathbf{F}_{1k}(\tau + \alpha_k, \nu_0, \nu_{01}, \ldots, \nu_{0n})] \tag{3.11}$$

The equations for determining the parameters of the generating solution can therefore be written as

$$P_k = P_{k0} + \sum_{j=1}^{n} P_{kj} \qquad (k = 1, 2, \ldots, n) \tag{3.12}$$
$$P_{kj} = \langle (Y_{k1}^{(2)})_0\, \mathbf{v}_{j0}\rangle$$
$$P_{k0} = \langle (Y_{k0}^{(2)})_0\rangle - \beta \langle (Y_k^{(1)})_0 (X_k^{(1)})_0\rangle + \beta \Big\langle U_k \Big(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\Big)_0\Big\rangle$$

In this case from (2.3) we have

$$\zeta_k = \frac{\partial \mathbf{v}_{0k}}{\partial \nu_0} \qquad (k = 1, 2, \ldots, n) \tag{3.13}$$

consequently

$$\Big\langle \Big(\frac{\partial Y_k^{(2)}}{\partial \mathbf{v}}\Big)_0 \zeta_j \Big\rangle = \Big\langle (Y_{k1}^{(2)})_0 \frac{\partial \mathbf{v}_{0j}}{\partial \nu_0}\Big\rangle = \frac{\partial P_{kj}}{\partial \nu_0} \tag{3.14}$$

Finally, the conditions of stability of the second group have the form

$$\sum_{k,j=1}^{n}\left\{\left[\beta\frac{\partial P_k}{\partial \nu_j}+\beta\frac{\partial R_k^{(2)}}{\partial \alpha_j}+\beta\frac{\partial P_{kj}}{\partial \nu_0}\right]a_{jr}a_{kr}^{*}+\right.$$
$$\left.+\left[\beta^3\langle(Y_k^{(1)})_0(X_k^{(1)})_0\rangle-\beta^3\left\langle\left(\frac{\partial Y_k^{(1)}}{\partial \omega_k}\right)_0 U_k\right\rangle a_{kr}a_{kr}^{*}\right\}<0 \qquad (3.15)$$

Assume now that the functions $Y_k^{(2)}\quad(\varphi_1,\ldots,\varphi_n,\omega_1,\ldots,\omega_n,\mathbf{v})$ linear with respect to the coordinates of the supporting system, have the form

$$Y_k^{(2)}(\varphi_1,\ldots,\varphi_n,\omega_1,\ldots,\omega_n,\mathbf{v})=Y_{k\circ}^{(2)}(\varphi_1,\ldots,\varphi_n,\omega_1,\ldots,\omega_n)+$$
$$+\sum_{k=1}^{m}Y_{ki}^{(2)}(\varphi_k,\omega_k)\,\xi_i \qquad (3.16)$$

where m of the so called reverse influence parameters [4, 5] ξ_i are connected with the coordinate vector $\mathbf{v}$ by the relations $\xi_i=(\mathbf{v},\ \mathbf{q}_i)$, $\mathbf{q}_i$ being constant vectors in the configurational space of the supporting system and the brackets denoting scalar products.

Assume, in addition, that the vector functions $\mathbf{F}_{1k}$ can be written in the form

$$\mathbf{F}_{1k}=F_{1k}(\varphi_k,\ \omega_k)\ \mathbf{q}_k \qquad (3.17)$$

The equation of motion of the supporting system can now be written as

$$\dot{\mathbf{v}}=A\mathbf{v}+\sum_{k=1}^{m}F_{1k}(\varphi_k,\omega_k)\,\mathbf{q}_k+\mu\ldots \qquad (3.18)$$

In the generating approximation we have

$$\mathbf{v}_0=\sum_{k=1}^{m}\mathbf{v}_{0k},\qquad \dot{\mathbf{v}}_{0k}=A\mathbf{v}_{0k}+F_{1k}(\tau+\alpha_k,\nu_k)\,\mathbf{q}_k \qquad (3.19)$$

Next expand the periodic functions F_{1k} into Fourier series

$$F_{1k}=\sum_{\rho=0}^{\infty}F_{1k}^{(\rho)}(\nu)\cos\left[\rho(\nu t+\alpha_k)+\Theta_k^{(\rho)}(\nu)\right] \qquad (3.20)$$

and seek the vectors $\mathbf{v}_{0k}$ in the form of series whose components satisfy the equations

$$\dot{\mathbf{v}}_k^{(\rho)}=A\mathbf{v}_k^{(\rho)}+F_{1k}^{(\rho)}\cos\left[\rho(\nu t+\alpha_k)+\Theta_k^{(\rho)}\right]\mathbf{q}_k\quad(\rho=0,1,\ldots) \qquad (3.21)$$

The latter possess solutions of the form

$$\mathbf{v}_k^{(\rho)}=F_{1k}^{(\rho)}(\nu)\ \mathbf{v}_{k*}^{(\rho)}\qquad(k=1,2,\ldots,m) \qquad (3.22)$$

where

$$\mathbf{v}_{j*}^{(\rho)}=\mathbf{v}_{k1}^{(\rho)}\cos\ \{\rho(\nu t+\alpha_k)+\Theta_k^{(\rho)}(\nu)\}+\mathbf{v}_{k2}^{(\rho)}\sin\{\rho(\nu t+\alpha_k)+\Theta_k^{(\rho)}(\nu)\} \qquad (3.23)$$

Here $\mathbf{v}_{k1}^{(\rho)}$ and $\mathbf{v}_{k2}^{(\rho)}$ are certain functions of the synchronous frequency.

Now introduce the quantities $K_{ij}^{(\rho)}(\nu)$ and $\Psi_{ij}^{(\rho)}(\nu)\quad(i,\ j=1,2,\ldots,m,\quad\rho=0,1,\ldots)$ defined from the solution of the linear problem on forced oscillations of the supporting system acted upon by prescribed harmonic forces as follows:

$$(\mathbf{v}_{i*}^{(\rho)},\ \mathbf{q}_i)=K_{ij}^{(\rho)}\cos\ [\rho(\nu t+\alpha_j)+\Theta_j^{(\rho)}(\nu)-\Psi_{ji}^{(\rho)}(\nu)] \qquad (3.24)$$

where

$$\dot{\mathbf{v}}_{j*}^{(\rho)}=A\mathbf{v}_{j*}^{(\rho)}+\cos\ \rho\ (\nu t+\alpha_j)\ \mathbf{q}_j$$

According to (3.22) and (3.24) we have

$$\xi_i=\sum_{j=1}^{m}\sum_{\rho=0}^{\infty}F_{1j}^{(\rho)}K_{ji}^{(\rho)}\cos\left[\rho(\nu t+\alpha_j)+\Theta_j^{(\rho)}(\nu)-\Psi_{ji}^{(\rho)}(\nu)\right] \qquad (3.25)$$

Equations defining the parameters of the generating solution in the given particular case will now become

$$P_k = P_{k0} + \sum_{i,\, j=1}^{m} \sum_{\rho=0}^{\infty} P_{kij}^{(\rho)} = 0 \qquad (3.26)$$

where

$$P_{kij}^{(\rho)} = \langle (Y_{ki}^{(2)})_0 \cos[\rho(\tau + \alpha_j) + \Theta_j^{(\rho)} - \Psi_{jk}^{(\rho)}] \rangle\, F_{1i}^{(\rho)}(\nu_0)\, K_{jk}^{(\rho)}(\nu_0) \qquad (3.27)$$

$$P_{k0} = \langle (Y_{k0}^{(2)})_0 \rangle - \beta \langle (Y_k^{(1)})_0 (X_k^{(1)})_0 \rangle + \beta \left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \omega_k} \right)_0 U_k \right\rangle$$

The only quantities dependent on the characteristics of the oscillatory system in (3.26) are the harmonic influence and phase coefficients. Consequently, once the equations have been set-up for a certain system of objects, they can be used to investigate the synchronization problems with any supporting system. The same applies to the conditions of stability based on these equations.

4. A system conservative in the first approximation. It has been shown previously that in this case the solutions can obviously be expanded in the powers of μ. We now consider this case in more detail.

Suppose the first $2n$ equations of the initial system correspond to the system conservative to within the μ^2-order terms

$$\varphi_k^{\cdot} = \omega_k + \mu X_k^{(1)}(\varphi_k, \omega_k) + \mu^2 \ldots = \frac{\partial H_k}{\partial \omega_k} + \mu^2 \ldots$$

$$\omega_k^{\cdot} = \mu Y_k^{(1)}(\varphi_k, \omega_k) + \mu^2 \ldots = -\frac{\partial H_k}{\partial \varphi_k} + \mu^2 \ldots \qquad (k = 1, 2, \ldots, n)$$

$$\mathbf{v}^{\cdot} = A\mathbf{v} + \mathbf{F}_1(\varphi_1, \ldots, \varphi_n, \omega_1, \ldots, \omega_n) + \mu \ldots \qquad (4.1)$$

Here the vectors $\mathbf{v}$, $\mathbf{F}_1$ as well as the functions $X_k^{(1)}$, $Y_k^{(1)}$ satisfy the conditions given in Sect. 1. The quantity $H_k(\varphi_k, \omega_k)$ represents a partial Hamiltonian of the kth object [1].

Under these conditions the following equations hold:

$$\beta \left\langle \left(\frac{\partial Y_k^{(1)}}{\partial \omega_k} \right)_0 U_k \right\rangle - \beta \langle (Y_k^{(1)})_0 (X_k^{(1)})_0 \rangle = 0 \qquad (4.2)$$

As a result it follows that the equations for the parameters of the generating solutions and the conditions of stability are similar to the corresponding relations given in [1].

The author thanks K. Sh. Khozhdaev for help given in the course of preparation of this paper.

BIBLIOGRAPHY

1. Nagaev, R. F. and Khozhdaev, K. Sh., Synchronous motions in a system of objects with supporting constraints. PMM Vol. 31, №4, 1967.
2. Malkin, I. G., Certain Problems of the Theory of Nonlinear Oscillations. M., Gostekhizdat, 1956.
3. Kushul', M. Ia., On the quasiharmonic systems close to systems with constant coefficients, in which pure imaginary roots of the fundamental equations have nonsimple elementary divisors. PMM Vol. 22, №4, 1958.
4. Khozhdaev, K. Sh., On excitation of vibrations. Inzh. zh. MTT, №2, 1968.
5. Nagaev, R. F., Synchronization of finite-dimensional "force" generators. PMM Vol. 32, №5, 1968.

Translated by L. K.